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Iterative algorithms for variational inequality and equilibrium problems with applications

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Abstract In this paper, we introduce an iterative method for finding a common element of the set of solutions of equilibrium problems, of the set of variational inequalities and of the set of common fixed points of an infinite family of nonexpansive mappings in the framework of real Hilbert spaces. Strong convergence of the proposed iterative algorithm is obtained. As an application, we utilize the main results which improve the corresponding results announced in Chang et al. (Nonlinear Anal, 70:3307–3319, 2009), Colao et al. (J Math Anal Appl, 344:340–352, 2008), Plubtieng and Punpaeng (Appl Math Comput, 197:548–558, 2008) to study the optimization problem.

Keywords Equilibrium problem · Variational inequality · Optimization · Nonexpansive mapping

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1 Introduction and preliminaries

Let H be a real Hilbert space, whose inner product and norm are denoted by $\langle\cdot,\cdot\rangle$ and $\|\cdot\|$, respectively. Let C be a nonempty closed convex subset of H and P_C the projection of H onto C.

Let f, S, A, B, T be five nonlinear mappings. Recall the following definitions.

(1) $f : H \to H$ is said to be contractive if there exists an $\alpha \in (0,1)$ such that

$$\|f(x) - f(y)\| \le \alpha\|x - y\|, \quad \forall x, y \in C.$$

(2) $S : C \to C$ is said to be nonexpansive if

$$\|Sx - Sy\| \le \|x - y\|, \quad \forall x, y \in C.$$

Throughout this paper, we use $F(S)$ to denote the set of fixed points of the mapping S. We also remark that if the subset C is nonempty bounded closed and convex then $F(S) \neq \emptyset$.

(3) $A : C \to H$ is said to be monotone if

$$\langle x - y, Ax - Ay\rangle \ge 0, \quad \forall x, y \in C.$$

(4) $A : C \to H$ is said to be inverse-strongly monotone if there exists $\delta > 0$ such that

$$\langle x - y, Ax - Ay\rangle \ge \delta\|Ax - Ay\|^2, \quad \forall x, y \in C.$$

Such a mapping A is also called δ-inverse-strongly monotone; see, for instance, [1,9]. We know that if $S : C \to C$ is nonexpansive, then $A = I - S$ is $\frac{1}{2}$-inverse-strongly monotone; see [1,23] for more details.

(5) $B : C \to H$ is said to be strongly positive if there is a constant $\bar{\gamma} > 0$ such that

$$\langle Bx, x\rangle \ge \bar{\gamma}\|x\|^2, \quad \forall x \in C.$$

(6) A set-valued mapping $T : H \to 2^H$ is called monotone if $f \in Tx$ and $g \in Ty$ imply that $\langle x - y, f - g\rangle \ge 0$ for all $x, y \in H$. A monotone mapping $T : H \to 2^H$ is maximal if the graph of $G(T)$ of T is not properly contained in the graph of any other monotone mapping. It is known that a monotone mapping T is maximal if and only if for $(x, f) \in H \times H$, $\langle x - y, f - g\rangle \ge 0$ for every $(y, g) \in G(T)$ implies $f \in Tx$. Let A be a monotone map of C into H and let $N_C v$ be the normal cone to C at $v \in C$, i.e., $N_C v = \{w \in H : \langle v - u, w\rangle \ge 0,\ \forall u \in C\}$ and define

$$Tv = \begin{cases} Av + N_C v, & v \in C \\ \emptyset, & v \notin C. \end{cases} \tag{Δ}$$

Then T is maximal monotone and $0 \in Tv$ if and only if $\langle Av, u - v\rangle \ge 0, \forall u \in C$; see [18] for more details.

Recall that the classical variational inequality is to find $u \in C$ such that

$$\langle Au, v - u\rangle \ge 0, \quad \forall v \in C. \tag{1.1}$$

In this paper, we use $VI(C, A)$ to denote the solution set of the variational inequality (1.1). For given $z \in H$ and $u \in C$, we see that the following inequality holds

$$\langle u - z, v - u\rangle \ge 0, \quad \forall v \in C,$$

if and only if $u = P_C z$. It is known that projection operator P_C is nonexpansive. It is also known that P_C satisfies

$$\langle x - y, P_C x - P_C y\rangle \geq \|P_C x - P_C y\|^2, \quad \forall x, y \in C. \tag{1.2}$$

One can see that the variational inequality (1.1) is equivalent to a fixed point problem. An element $u \in C$ is a solution of the variational inequality (1.1) if and only if $u \in C$ is a fixed point of the mapping $P_C(I - \lambda A)u$, where $\lambda > 0$ is a constant and I is the identity mapping. This alternative equivalent formulation has played a significant role in the studies of the variational inequalities and related optimization problems.

Iterative methods for nonexpansive mappings have recently been applied to solve convex minimization problems; see, e.g., [8,11,27,28,31] and the references therein. A typical problem is to minimize a quadratic function over the set of the fixed points of a nonexpansive mapping on a real Hilbert space H:

$$\min_{x \in F(S)} \frac{1}{2}\langle Bx, x\rangle - h(x), \tag{1.3}$$

where B is a linear bounded operator defined on H, $F(S)$ is the fixed point set of the nonexpansive mapping S and h is a potential function for γf (i.e., $h'(x) = \gamma f(x)$ for $x \in H$).

Recently, Marino and Xu [11] studied the following iterative scheme

$$x_0 \in H, \quad x_{n+1} = (I - \alpha_n B)Sx_n + \alpha_n \gamma f(x_n), \quad n \geq 0,$$

where f is a contraction defined on H, B is a strongly positive linear bounded operator and S is a nonexpansive mapping with a fixed point. They proved that the sequence $\{x_n\}$ generated by the above iterative scheme converges strongly to the unique solution of the variational inequality

$$\langle (B - \gamma f)x^*, x - x^*\rangle \geq 0, \quad \forall x \in F(S),$$

which is the optimality condition for the minimization problem (1.3).

Let F be a bifunction of $C \times C$ into $\mathbb{R}$, where $\mathbb{R}$ is the set of real numbers. We consider the following equilibrium problem:

$$\text{Find } u \in C \text{ such that } F(u, y) \geq 0, \quad \forall y \in C. \tag{1.4}$$

In this paper, the set of such $u \in C$ is denoted by $EP(F)$, i.e.,

$$EP(F) = \{u \in C : F(u, y) \geq 0, \quad \forall y \in C\}.$$

Numerous problems in physics, optimization, and economics reduce to find a solution of (1.4); see, for instance, [2,7,10].

Throughout this paper, we assume that the bifunction $F : C \times C \to \mathbb{R}$ satisfies the following conditions:

(A1) $F(x, x) = 0$ for all $x \in C$;
(A2) F is monotone, i.e., $F(x, y) + F(y, x) \leq 0$ for all $x, y \in C$;
(A3) for each $x, y, z \in C$,

$$\limsup_{t \downarrow 0} F(tz + (1 - t)x, y) \leq F(x, y);$$

(A4) for each $x \in C$, $y \mapsto F(x, y)$ is convex and lower semi-continuous.

Recently, Takahashi and Takahashi [22] introduced the following iterative method

$$\begin{cases} F(y_n, u) + \frac{1}{r_n}\langle u - y_n, y_n - x_n\rangle \geq 0, & \forall u \in C, \\ x_{n+1} = \alpha_n f(x_n) + (1-\alpha_n)Ty_n, & n \geq 1 \end{cases}$$

for approximating a common element of the set of fixed points of a single nonexpansive mapping and the set of solutions of the equilibrium problem (1.4). They obtained a strong convergence theorem in a real Hilbert space.

Subsequently, many authors studied the problem of finding a common element of the set of fixed points nonexpansive mappings, the set of solutions of variational inequalities and the set of solutions of equilibrium problems; see [3–6,13–17,20,22,24,25,29,30] for more details.

Recently, Colao et al. [6] studied the equilibrium problem (1.4) and a W-mapping, which was generated by a finite family of nonexpansive mappings; see [6] for more details, and prove the following theorem.

Theorem VMX *Let H be a Hilbert space, C a closed convex nonempty subset of H, $\{T_i\}_{i=1}^N$ a finite family of nonexpansive mappings from H into itself, $G : C \times C \to \mathbb{R}$ a bifunction, A a strongly positive bounded linear operator with coefficient $\bar{\gamma}$ and f an α-contraction on H for some $0 < \alpha < 1$. Moreover, let $\{\epsilon_n\}$ be a sequence in $(0, 1)$, $\{\lambda_{n,i}\}_{i=1}^N$ a sequence in $[a, b]$ with $0 < a \leq b < 1$, $\{r_n\}$ a sequence in $(0, \infty)$ and γ and β two real numbers such that $0 < \beta < 1$ and $0 < \gamma < \bar{\gamma}/\alpha$. Assume the bifuction G satisfies $(A1)$–$(A4)$ and $F \cap EP(G) \neq \emptyset$, where $F = \cap_{i=1}^N F(T_i)$, and the sequence $\{\epsilon_n\}$ satisfies*

(1) $\lim_{n\to\infty} \epsilon_n = 0$, *and* $\sum_{n=1}^{\infty} \epsilon_n = \infty$,
the sequence $\{r_n\}$ satisfies
(2) $\liminf_{n\to\infty} r_n > 0$, $\lim_{n\to\infty} r_n/r_{n+1} = 1$,
the finite family of sequence $\{\lambda_{n,i}\}_{i=1}^N$ satisfies
(3) $\lim_{n\to\infty} |\lambda_{n,i} - \lambda_{n-1,i}| = 0$, *for every $i \in \{1, 2, \ldots, N\}$.*
For every $n \in \mathbb{N}$, let W_n be the W-mapping generated by $T_1, \ldots, T_N$ and $\lambda_{n,1}, \ldots, \lambda_{n,N}$. If $\{x_n\}$ and $\{u_n\}$ are the sequences generated by $x_1 \in H$ and $\forall n \geq 1$.

$$\begin{cases} F(u_n, y) + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq 0, & \forall y \in H, \\ x_{n+1} = \alpha_n\gamma f(x_n) + \beta_n x_n + [(1-\beta_n)I - \alpha_n B]W_n u_n, \end{cases} \tag{1.5}$$

then both $\{x_n\}$ and $\{u_n\}$ converge strongly to $x^* \in F$, where x^* is an equilibrium point for F and is the unique solution of variational inequality,

$$\langle (A - \gamma f)x^*, x^* - x\rangle \leq 0, \quad \forall x \in F \cap EP(G).$$

Very recently, Chang, Lee and Chan [5] introduced a new iterative method for solving the variational inequality (1.1), fixed point problems of nonexpansive mappings and the equilibrium problem (1.4) in the framework of real Hilbert spaces. More precisely, they proved the following theorem.

Theorem CLC *Let H be a real Hilbert space, C be a nonempty closed convex subset of H, F be a bifunction satisfying the conditions $(A1)$–$(A4)$, $A : C \to H$ be an α-inverse-strongly monotone mapping, $\{S_i : C \to C\}$ be a family of infinitely nonexpansive mappings with $\mathcal{F} \cap VI(C, A) \cap EP(F) \neq \emptyset$, where $\mathcal{F} := \cap_{i=1}^{\infty} F(S_i)$ and $f : H \to H$ be a contraction mapping with a contractive constant $\xi \in (0, 1)$. Let $\{x_n\}$, $\{y_n\}\{k_n\}$ and $\{u_n\}$ be sequences defined by*

$$\begin{cases} F(u_n, y) + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq 0, \ \forall y \in C, \\ x_{n+1} = \alpha_n f(x_n) + \beta_n x_n + \gamma_n W_n k_n, \quad \forall n \geq 1, \\ k_n = P_C(y_n - \lambda_n A y_n), \\ y_n = P_C(u_n - \lambda_n A u_n), \end{cases} \tag{1.6}$$

where $\{W_n : C \to C\}$ *is the sequence defined by* (1.7), $\{\alpha_n\}$, $\{\beta_n\}$ *and* $\{\gamma_n\}$ *are sequences in* $[0, 1]$, $\{\lambda_n\}$ *is a sequence in* $[a, b] \subset (0, 2\alpha)$ *and* $\{r_n\}$ *is a sequence in* $(0, \infty)$. *If the following conditions are satisfied:*

(1) $\alpha_n + \beta_n + \gamma_n = 1$;
(2) $\lim_{n\to\infty} \alpha_n = 0$; $\sum_{n=1}^{\infty} \alpha_n = \infty$;
(3) $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$;
(4) $\liminf_{n\to\infty} r_n > 0$; $\sum_{n=1}^{\infty} |r_{n+1} - r_n| < \infty$;
(5) $\lim_{n\to\infty} |\lambda_{n+1} - \lambda_n| = 0$

then $\{x_n\}$ *and* $\{u_n\}$ *converge strongly to* $z \in \mathcal{F} \cap VI(C, A) \cap EP(F)$, *where* $z = P_{\mathcal{F}\cap VI(C,A)\cap EP(F)} f(z)$.

In this paper, motivated and inspired by the research going on in this direction, we introduce a general iterative method for finding a common element of the set of solutions of the equilibrium problem (1.4), the set of solutions of variational inequalities and the set of common fixed points of a family of nonexpansive mappings in the framework of real Hilbert spaces. The results presented in this paper improve and extend the corresponding results of Chang Lee and Chan [5], Ceng and Yao [3,4], Iiduka and Takahashi [9], Qin, Shang and Zhou [14], Su, Shang and Qin [20], Takahashi and Takahashi [22], Takahashi and Toyoda [24], Verma [26], Yao, Noor and Liou [29], Yao and Yao [30] and many others.

In order to prove our main results, we need the following definitions and lemmas.

A space X is said to satisfy Opial's condition [12] if for each sequence $\{x_n\}$ in X which converges weakly to point $x \in X$, we have

$$\liminf_{n\to\infty} \|x_n - x\| < \liminf_{n\to\infty} \|x_n - y\|, \quad \forall y \in X, y \neq x.$$

It is well known that [12] all Hilbert spaces and $l_p (p > 1)$ satisfy Opial's condition, while L_p does not unless $p = 2$.

Lemma 1.1 ([2]). *Let* C *be a nonempty closed convex subset of* H *ad* $F : C \times C \to \mathbb{R}$ *a bifunction satisfying* (A1)–(A4). *Then for any* $r > 0$ *and* $x \in H$, *there exists* $z \in C$ *such that*

$$F(z, y) + \frac{1}{r}\langle y - z, z - x\rangle \geq 0, \quad \forall y \in C.$$

Lemma 1.2 ([7]). *Suppose that all the conditions in Lemma* 1.1 *are satisfied. For* $r > 0$ *and* $x \in H$, *define a mapping* $T_r : H \to C$ *as follows:*

$$T_r(x) = \left\{ z \in C : F(z, y) + \frac{1}{r}\langle y - z, z - x\rangle \geq 0, \quad \forall y \in C \right\}$$

for $z \in H$. *Then the following conclusions hold:*

(1) T_r *is single-valued;*
(2) T_r *is firmly nonexpansive, i.e., for any* $x, y \in H$,

$$\|T_r x - T_r y\|^2 \leq \langle T_r x - T_r y, x - y\rangle;$$

(3) $F(T_r) = EP(F)$;
(4) $EP(F)$ *is closed and convex.*

Lemma 1.3 *Let H be a real Hilbert space. The following inequality holds*

$$\|x + y\|^2 \leq \|x\|^2 + 2\langle y, x + y\rangle, \quad x, y \in H.$$

Lemma 1.4 ([27]). *Assume that $\{\alpha_n\}$ is a sequence of nonnegative real numbers such that*

$$\alpha_{n+1} \leq (1 - \gamma_n)\alpha_n + \delta_n,$$

where $\{\gamma_n\}$ is a sequence in $(0, 1)$ and $\{\delta_n\}$ is a sequence such that

(1) $\sum_{n=1}^{\infty} \gamma_n = \infty$;
(2) $\limsup_{n\to\infty} \delta_n/\gamma_n \leq 0$ *or* $\sum_{n=1}^{\infty} |\delta_n| < \infty$.

Then $\lim_{n\to\infty} \alpha_n = 0$.

Definition 1.5 ([21]). Let $\{S_i : C \to C\}$ be a family of infinitely nonexpansive mappings and $\{\gamma_i\}$ be a nonnegative real sequence with $0 \leq \gamma_i < 1, \forall i \geq 1$. For $n \geq 1$ define a mapping $W_n : C \to C$ as follows:

$$\begin{aligned}
&U_{n,n+1} = I,\\
&U_{n,n} = \gamma_n S_n U_{n,n+1} + (1 - \gamma_n)I,\\
&U_{n,n-1} = \gamma_{n-1} S_{n-1} U_{n,n} + (1 - \gamma_{n-1})I,\\
&\vdots\\
&U_{n,k} = \gamma_k S_k U_{n,k+1} + (1 - \gamma_k)I,\\
&U_{n,k-1} = \gamma_{k-1} S_{k-1} U_{n,k} + (1 - \gamma_{k-1})I,\\
&\vdots\\
&U_{n,2} = \gamma_2 S_2 U_{n,3} + (1 - \gamma_2)I,\\
&W_n = U_{n,1} = \gamma_1 S_1 U_{n,2} + (1 - \gamma_1)I.
\end{aligned} \tag{1.7}$$

Such a mapping W_n is nonexpansive from C to C and it is called a W-mapping generated by $S_n, S_{n-1}, \ldots, S_1$ and $\gamma_n, \gamma_{n-1}, \ldots, \gamma_1$.

Lemma 1.6 ([21]). *Let C be a nonempty closed convex subset of a Hilbert space H, $\{S_i : C \to C\}$ a family of infinitely nonexpansive mappings with $\cap_{i=1}^{\infty} F(S_i) \neq \emptyset$ and $\{\gamma_i\}$ a real sequence such that $0 < \gamma_i \leq l < 1, \forall i \geq 1$. Then*

(1) *W_n is nonexpansive and $F(W_n) = \cap_{i=1}^{n} F(S_i)$, for each $n \geq 1$;*
(2) *for each $x \in C$ and for each positive integer k, the limit $\lim_{n\to\infty} U_{n,k}$ exists.*
(3) *the mapping $W : C \to C$ defined by*

$$Wx := \lim_{n\to\infty} W_n x = \lim_{n\to\infty} U_{n,1} x, \quad x \in C,$$

is a nonexpansive mapping satisfying $F(W) = \cap_{i=1}^{\infty} F(S_i)$ and it is called the W-mapping generated by $S_1, S_2, \ldots$ and $\gamma_1, \gamma_2, \ldots$.

Lemma 1.7 ([5]). *Let C be a nonempty closed convex subset of a Hilbert space H, $\{S_i : C \to C\}$ a family of infinitely nonexpansive mappings with $\cap_{i=1}^{\infty} F(S_i) \neq \emptyset$ and $\{\gamma_i\}$ a real sequence such that $0 < \gamma_i \leq l < 1, \forall i \geq 1$. If K is any bounded subset of C, then*

$$\lim_{n\to\infty} \sup_{x\in K} \|Wx - W_n x\| = 0.$$

Throughout this paper, we always assume that $0 < \gamma_i \leq l < 1, \forall i \geq 1$.

Lemma 1.8 ([19]). *Let $\{x_n\}$ and $\{y_n\}$ be bounded sequences in a real Hilbert space H and let $\{\beta_n\}$ be a sequence in $(0, 1)$ with $0 < \liminf_{n\to\infty} \beta_n \le \limsup_{n\to\infty} \beta_n < 1$. Suppose that $x_{n+1} = (1 - \beta_n)y_n + \beta_n x_n$ for all $n \ge 0$ and*

$$\limsup_{n\to\infty} (\|y_{n+1} - y_n\| - \|x_{n+1} - x_n\|) \le 0.$$

Then $\lim_{n\to\infty} \|y_n - x_n\| = 0$.

2 Main results

Theorem 2.1 *Let C be a nonempty closed convex subset of a Hilbert space H and F a bifunction from $C \times C$ to $\mathbb{R}$ which satisfies (A1)–(A4). Let A_j be a δ_j-inverse-strongly monotone mapping of C into H for each $j = 1, 2$, $\{S_i : C \to C\}$ a family of infinitely nonexpansive mappings with $\Omega := FP \cap EP(F) \cap VI \neq \emptyset$, where $FP = \cap_{i=1}^{\infty} F(S_i)$ and $VI = VI(C, A_1) \cap VI(C, A_2)$. Let f be a contraction of H into itself with the contractive coefficient $\alpha(0 < \alpha < 1)$, $B : C \to H$ a strongly positive linear bounded self-joint operator with the coefficient $\bar{\gamma} > 0$. Assume that $0 < \gamma < \bar{\gamma}/\alpha$. Let $\{x_n\}$ be a sequence generated by $x_1 \in H$ and*

$$\begin{cases} F(u_n, y) + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \ge 0, \quad \forall y \in C, \\ z_n = P_C(u_n - \lambda_n A_2 u_n), \\ y_n = P_C(z_n - \eta_n A_1 z_n), \\ x_{n+1} = \alpha_n \gamma f(x_n) + \beta_n x_n + [(1 - \beta_n)I - \alpha_n B]W_n y_n, \quad \forall n \ge 1, \end{cases} \tag{2.1}$$

where $\{W_n : C \to C\}$ is the sequence defined by (1.7), $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in $[0, 1]$ and $\{r_n\}$, $\{\lambda_n\}$ and $\{\eta_n\}$ are positive number sequences. Assume that the above control sequences satisfy the following restrictions

(C1) $0 < a \le \beta_n \le b < 1, 0 < c \le \eta_n \le d < 2\delta_1, 0 < c' \le \lambda_n \le d' < 2\delta_2, \forall n \ge 1$;
(C2) $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=1}^{\infty} \alpha_n = \infty$;
(C3) $\liminf_{n\to\infty} r_n > 0, \lim_{n\to\infty} r_n/r_{n+1} = 1$;
(C4) $\lim_{n\to\infty}(\lambda_n - \lambda_{n+1}) = \lim_{n\to\infty}(\eta_n - \eta_{n+1}) = 0$.

Then the sequence $\{x_n\}$ converges strongly to $z \in \Omega$, which solves uniquely the following variational inequality

$$\langle (B - \gamma f)z, z - x\rangle \le 0, \quad \forall x \in \Omega. \tag{2.2}$$

Equivalently, we have $P_\Omega(I - B + \gamma f)z = z$.

Proof The uniqueness of the solution of the variational inequality (2.2) is a consequence of the strong monotonicity of $B - \gamma f$. Indeed, suppose that $z_1 \in \Omega$ and $z_2 \in \Omega$ both are solutions to (2.2). Then, we have

$$\langle (B - \gamma f)z_1, z_1 - z_2\rangle \le 0$$

and

$$\langle (B - \gamma f)z_2, z_2 - z_1\rangle \le 0.$$

Adding up the two inequalities, we see that

$$\langle (B - \gamma f)z_1 - (B - \gamma f)z_2, z_1 - z_2\rangle \le 0.$$

The strong monotonicity of $B - \gamma f$ (see Lemma 2.3 of [11]) implies that $z_1 = z_2$ and the uniqueness is proved. Below we use z to denote the unique solution of (2.2).

Next, we show, for each $n \geq 1$, that the mappings $I - \eta_n A_1$ and $I - \lambda_n A_2$ are nonexpansive. Indeed, for $\forall x, y \in C$, from the condition (C1) one has

$$\begin{aligned}
&\|(I - \eta_n A_1)x - (I - \eta_n A_1)y\|^2 \\
&\quad = \|(x - y) - \eta_n(A_1 x - A_1 y)\|^2 \\
&\quad = \|x - y\|^2 - 2\eta_n \langle x - y, A_1 x - A_1 y\rangle + \eta_n^2 \|A_1 x - A_1 y\|^2 \\
&\quad \leq \|x - y\|^2 - 2\eta_n \delta_1 \|A_1 x - A_1 y\|^2 + \eta_n^2 \|A_1 x - A_1 y\|^2 \\
&\quad = \|x - y\|^2 + \eta_n(\eta_n - 2\delta_1)\|A_1 x - A_1 y\|^2 \\
&\quad \leq \|x - y\|^2,
\end{aligned}$$

which implies the mapping $I - \eta_n A_1$ is nonexpansive, so is $I - \lambda_n A_2$ for each $n \geq 1$. Note that u_n can be re-written as $u_n = T_{r_n} x_n$ for each $n \geq 1$. Take $x^* \in \Omega$. Noticing that $x^* = T_{r_n} x^*$, we have

$$\|u_n - x^*\| = \|T_{r_n} x_n - T_{r_n} x^*\| \leq \|x_n - x^*\|. \tag{2.3}$$

On the other hand, we have

$$\begin{aligned}
\|z_n - x^*\| &= \|P_C(u_n - \lambda_n A_2 u_n) - P_C(x^* - \lambda_n A_2 x^*)\| \\
&\leq \|(u_n - \lambda_n A_2 u_n) - (x^* - \lambda_n A_2 x^*)\| \\
&\leq \|u_n - x^*\|.
\end{aligned} \tag{2.4}$$

It follows from (2.3) and (2.4) that

$$\|z_n - x^*\| \leq \|x_n - x^*\|, \tag{2.5}$$

which yields that

$$\begin{aligned}
\|y_n - x^*\| &= \|P_C(z_n - \eta_n A_1 z_n) - P_C(x^* - \eta_n A_1 x^*)\| \\
&\leq \|(z_n - \eta_n A_1 z_n) - (x^* - \eta_n A_1 x^*)\| \\
&\leq \|z_n - x^*\| \\
&\leq \|x_n - x^*\|.
\end{aligned} \tag{2.6}$$

Note that from the condition (C2), we may assume, without loss of generality, that $\alpha_n \leq (1 - \beta_n)\|B\|^{-1}$ for all $n \geq 1$. Since B is a strongly positive linear bounded self-adjoint operator on C, we have

$$\|B\| = \sup\{|\langle Bx, x\rangle| : x \in C, \|x\| = 1\},$$

Now for $x \in C$ with $\|x\| = 1$, we see that

$$\begin{aligned}
\langle((1 - \beta_n)I - \alpha_n B)x, x\rangle &= 1 - \beta_n - \alpha_n \langle Bx, x\rangle \\
&\geq 1 - \beta_n - \alpha_n \|B\| \\
&\geq 0,
\end{aligned}$$

that is, $(1 - \beta_n)I - \alpha_n A$ is positive. It follows that

$$\begin{aligned}
\|(1 - \beta_n)I - \alpha_n B\| &= \sup\{\langle((1 - \beta_n)I - \alpha_n B)x, x\rangle : x \in C, \|x\| = 1\} \\
&= \sup\{1 - \beta_n - \alpha_n \langle Bx, x\rangle : x \in C, \|x\| = 1\} \\
&\leq 1 - \beta_n - \alpha_n \bar{\gamma}.
\end{aligned} \tag{2.7}$$

From (2.1), (2.6) and (2.7), we arrive at

$$\begin{aligned}
\|x_{n+1} - x^*\| &= \|\alpha_n \gamma f(x_n) + \beta_n x_n + [(1-\beta_n)I - \alpha_n B]W_n y_n - x^*\| \\
&\le \alpha_n \|\gamma f(x_n) - Bx^*\| + \beta_n \|x_n - x^*\| + \|(1-\beta_n)I - \alpha_n B\| \|W_n y_n - x^*\| \\
&\le \alpha_n \|\gamma f(x_n) - Bx^*\| + \beta_n \|x_n - x^*\| + (1 - \beta_n - \alpha_n \bar{\gamma})\|W_n y_n - x^*\| \\
&\le \alpha_n \|\gamma f(x_n) - \gamma f(x^*)\| + \alpha_n \|\gamma f(x^*) - Bx^*\| + \beta_n \|x_n - x^*\| \\
&\quad + (1 - \beta_n - \alpha_n \bar{\gamma})\|y_n - x^*\| \\
&\le \alpha \alpha_n \gamma \|x_n - x^*\| + \alpha_n \|\gamma f(x^*) - Bx^*\| + \beta_n \|x_n - x^*\| \\
&\quad + (1 - \beta_n - \alpha_n \bar{\gamma})\|x_n - x^*\| \\
&= [1 - \alpha_n(\bar{\gamma} - \alpha\gamma)]\|x_n - x^*\| + \alpha_n \|\gamma f(x^*) - Bx^*\|.
\end{aligned}$$

By simple inductions, we obtain that

$$\|x_n - x^*\| \le \max\left\{\|x_1 - x^*\|, \frac{\|\gamma f(x^*) - Bx^*\|}{\bar{\gamma} - \alpha\gamma}\right\},$$

which gives that the sequence $\{x_n\}$ is bounded, so are $\{y_n\}$, $\{z_n\}$ and $\{u_n\}$. Without loss of generality, we can assume that there exists a bounded set $K \subset C$ such that

$$y_n, z_n, u_n \in K, \quad \forall n \ge 1. \tag{2.8}$$

Notice that $u_{n+1} = T_{r_{n+1}} x_{n+1}$ and $u_n = T_{r_n} x_n$. We see from Lemma 1.2 that

$$F(u_{n+1}, y) + \frac{1}{r_{n+1}} \langle y - u_{n+1}, u_{n+1} - x_{n+1} \rangle \ge 0, \quad \forall y \in C, \tag{2.9}$$

and

$$F(u_n, y) + \frac{1}{r_n} \langle y - u_n, u_n - x_n \rangle \ge 0, \quad \forall y \in C. \tag{2.10}$$

Let $y = u_n$ in (2.9) and $y = u_{n+1}$ in (2.10). By adding up these two inequalities and using the condition (A2), we obtain that

$$\left\langle u_{n+1} - u_n, \frac{u_n - x_n}{r_n} - \frac{u_{n+1} - x_{n+1}}{r_{n+1}} \right\rangle \ge 0.$$

Hence, we have

$$\left\langle u_{n+1} - u_n, u_n - u_{n+1} + u_{n+1} - x_n - \frac{r_n}{r_{n+1}}(u_{n+1} - x_{n+1}) \right\rangle \ge 0.$$

This implies that

$$\begin{aligned}
\|u_{n+1} - u_n\|^2 &\le \left\langle u_{n+1} - u_n, x_{n+1} - x_n + \left(1 - \frac{r_n}{r_{n+1}}\right)(u_{n+1} - x_{n+1}) \right\rangle \\
&\le \|u_{n+1} - u_n\| \left[\|x_{n+1} - x_n\| + \left|1 - \frac{r_n}{r_{n+1}}\right| \|u_{n+1} - x_{n+1}\|\right].
\end{aligned}$$

It follows that

$$\begin{aligned}
\|u_{n+1} - u_n\| &\le \|x_{n+1} - x_n\| + \left|1 - \frac{r_n}{r_{n+1}}\right| \|u_{n+1} - x_{n+1}\| \\
&\le \|x_{n+1} - x_n\| + M_1 \left|1 - \frac{r_n}{r_{n+1}}\right|,
\end{aligned} \tag{2.11}$$

where M_1 is an appropriate constant such that $M_1 = \sup_{n\geq 1}\{\|u_n - x_n\|\}$. On the other hand, from (2.1) and the nonexpansivity of the mapping $I - \lambda_n A_2$, $\forall n \geq 1$, we see that

$$\begin{aligned}\|z_{n+1} - z_n\| &= \|P_C(u_{n+1} - \lambda_{n+1}A_2u_{n+1}) - P_C(u_n - \lambda_n A_2 u_n)\|\\ &\leq \|u_{n+1} - \lambda_{n+1}A_2u_{n+1} - (u_n - \lambda_n A_2u_n)\|\\ &= \|(I - \lambda_{n+1}A_2)u_{n+1} - (I - \lambda_{n+1}A_2)u_n + (\lambda_n - \lambda_{n+1})A_2u_n\|\\ &\leq \|u_{n+1} - u_n\| + |\lambda_n - \lambda_{n+1}|\|A_2u_n\|. \end{aligned} \tag{2.12}$$

Substituting (2.11) into (2.12), we arrive at

$$\|z_{n+1} - z_n\| \leq \|x_{n+1} - x_n\| + M_2\left(\left|1 - \frac{r_n}{r_{n+1}}\right| + |\lambda_n - \lambda_{n+1}|\right), \tag{2.13}$$

where M_2 is an appropriate constant such that $M_2 = \max\{\sup_{n\geq 1}\{\|A_2u_n\|\}, M_1\}$.

On the other hand, we have

$$\begin{aligned}\|y_{n+1} - y_n\| &= \|P_C(z_{n+1} - \eta_{n+1}A_1z_{n+1}) - P_C(z_n - \eta_n A_1z_n)\|\\ &\leq \|z_{n+1} - \eta_{n+1}A_1z_{n+1} - (z_n - \eta_n A_1z_n)\|\\ &= \|(I - \eta_{n+1}A_1)z_{n+1} - (I - \eta_{n+1}A_1)z_n + (\eta_n - \eta_{n+1})A_1z_n\|\\ &\leq \|z_{n+1} - z_n\| + |\eta_n - \eta_{n+1}|\|A_1z_n\|. \end{aligned} \tag{2.14}$$

Substituting (2.13) into (2.14), we obtain that

$$\|y_{n+1} - y_n\| \leq \|x_{n+1} - x_n\| + M_3\left(\left|1 - \tfrac{r_n}{r_{n+1}}\right| + |\lambda_n - \lambda_{n+1}| + |\eta_n - \eta_{n+1}|\right), \tag{2.15}$$

where M_3 is an appropriate constant such that $M_3 \geq \max\{M_2, \sup_{n\geq 1}\{\|A_1z_n\|\}\}$. Letting

$$x_{n+1} = (1 - \beta_n)v_n + \beta_n x_n, \quad \forall n \geq 1, \tag{2.16}$$

we see that

$$\begin{aligned} v_{n+1} - v_n &= \frac{\alpha_{n+1}\gamma f(x_{n+1}) + [(1 - \beta_{n+1})I - \alpha_{n+1}B]W_{n+1}y_{n+1}}{1 - \beta_{n+1}}\\ &\quad - \frac{\alpha_n\gamma f(x_n) + [(1 - \beta_n)I - \alpha_n B]W_n y_n}{1 - \beta_n}\\ &= \frac{\alpha_{n+1}}{1 - \beta_{n+1}}[\gamma f(x_{n+1} - BW_n y_n] + W_{n+1}y_{n+1}\\ &\quad - \frac{\alpha_n}{1 - \beta_n}[\gamma f(x_n) - BW_n y_n] - W_n y_n. \end{aligned}$$

It follows that

$$\begin{aligned}\|v_{n+1} - v_n\| &\leq \frac{\alpha_{n+1}}{1 - \beta_{n+1}}\|\gamma f(x_{n+1} - BW_n y_n\| + \frac{\alpha_n}{1 - \beta_n}\|\gamma f(x_n) - BW_n y_n\|\\ &\quad + \|W_{n+1}y_{n+1} - W_n y_n\|. \end{aligned} \tag{2.17}$$

On the other hand, we have

$$\begin{aligned}\|W_{n+1}y_{n+1}-W_ny_n\| &= \|W_{n+1}y_{n+1}-Wy_{n+1}+Wy_{n+1}-Wy_n+Wy_n-W_ny_n\|\\ &\le \|W_{n+1}y_{n+1}-Wy_{n+1}\|+\|Wy_{n+1}-Wy_n\|+\|Wy_n-W_ny_n\|\\ &\le \sup_{x\in K}\{\|W_{n+1}x-Wx\|+\|Wx-W_nx\|\}+\|y_{n+1}-y_n\|,\end{aligned} \tag{2.18}$$

where K is the bounded subset of C defined by (2.8). Substituting (2.15) into (2.18), one arrives at

$$\begin{aligned}\|W_{n+1}y_{n+1}-W_ny_n\| \le{}& \sup_{x\in K}\{\|W_{n+1}x-Wx\|+\|Wx-W_nx\|\}+\|x_{n+1}-x_n\|\\ &+M_3\left(\left|1-\frac{r_n}{r_{n+1}}\right|+|\lambda_n-\lambda_{n+1}|+|\eta_n-\eta_{n+1}|\right),\end{aligned}$$

which combines with (2.17) yields that

$$\begin{aligned}&\|v_{n+1}-v_n\|-\|x_{n+1}-x_n\|\\ &\quad\le \frac{\alpha_{n+1}}{1-\beta_{n+1}}\|\gamma f(x_{n+1}-BW_ny_n\|+\frac{\alpha_n}{1-\beta_n}\|\gamma f(x_n)-BW_ny_n\|\\ &\quad+\sup_{x\in K}\{\|W_{n+1}x-Wx\|+\|Wx-W_nx\|\}\\ &\quad+M_3\left(\left|1-\frac{r_n}{r_{n+1}}\right|+|\lambda_n-\lambda_{n+1}|+|\eta_n-\eta_{n+1}|\right).\end{aligned}$$

It follows from the conditions (C2)–(C4) and Lemma 1.7 that

$$\limsup_{n\to\infty}(\|v_{n+1}-v_n\|-\|x_{n+1}-x_n\|)\le 0.$$

Hence, we obtain from Lemma 1.8 that

$$\lim_{n\to\infty}\|v_n-x_n\|=0.$$

From (2.16), we see that

$$\|x_{n+1}-x_n\|=(1-\beta_n)\|v_n-x_n\|.$$

It follows that

$$\lim_{n\to\infty}\|x_{n+1}-x_n\|=0. \tag{2.19}$$

Put $f_n=\gamma f(x_n)-BW_ny_n$, for $\forall n\ge 1$. For any $x^*\in\Omega$, we from Lemma 1.3 see that

$$\begin{aligned}\|x_{n+1}-x^*\|^2 &= \|\alpha_n\gamma f(x_n)+\beta_nx_n+[(1-\beta_n)I-\alpha_nB]W_ny_n-x^*\|^2\\ &= \|\alpha_n\gamma f(x_n)+\beta_nx_n+(1-\beta_n)W_ny_n-\alpha_nBW_ny_n-x^*\|^2\\ &= \|\alpha_n(\gamma f(x_n)-BW_ny_n)+[\beta_n(x_n-x^*)+(1-\beta_n)(W_ny_n-x^*)]\|^2\\ &\le \|\beta_n(x_n-x^*)+(1-\beta_n)(W_ny_n-x^*)\|^2+2\alpha_n\langle f_n,x_{n+1}-x^*\rangle\\ &\le \beta_n\|x_n-x^*\|^2+(1-\beta_n)\|W_ny_n-x^*\|^2+2\alpha_n\|f_n\|\|x_{n+1}-x^*\|\\ &\le \beta_n\|x_n-x^*\|^2+(1-\beta_n)\|y_n-x^*\|^2+2\alpha_nM_4^2,\end{aligned} \tag{2.20}$$

where M_4 is an appropriate constant such that

$$M_4 = \max\{\sup_{n\geq 1} \|f_n\|, \sup_{n\geq 1} \|x_n - x^*\|\}.$$

On the other hand, it follows from (2.1) that

$$\begin{aligned}\|y_n - x^*\|^2 &= \|P_C(z_n - \eta_n A_1 z_n) - x^*\|^2 \\ &\leq \|(I - \eta_n A_1)z_n - (I - \eta_n A_1)x^*\|^2 \\ &= \|(z_n - x^*) - \eta_n(A_1 z_n - A_1 x^*)\|^2 \\ &= \|z_n - x^*\|^2 - 2\eta_n\langle z_n - x^*, A_1 z_n - A_1 x^*\rangle + \eta_n^2\|A_1 z_n - A_1 x^*\|^2 \\ &\leq \|z_n - x^*\|^2 - 2\eta_n\delta_1\|A_1 z_n - A_1 x^*\|^2 + \eta_n^2\|A_1 z_n - A_1 x^*\|^2 \\ &= \|z_n - x^*\|^2 + \eta_n(\eta_n - 2\delta_1)\|A_1 z_n - A_1 x^*\|^2. \end{aligned} \tag{2.21}$$

Substituting (2.21) into (2.20), we arrive at

$$\begin{aligned}\|x_{n+1} - x^*\|^2 &\leq \beta_n\|x_n - x^*\|^2 + (1 - \beta_n)(\|z_n - x^*\|^2 \\ &\quad +\eta_n(\eta_n - 2\delta_1)\|A_1 z_n - A_1 x^*\|^2) + 2\alpha_n M_4^2 \\ &\leq \beta_n\|x_n - x^*\|^2 + (1 - \beta_n)(\|x_n - x^*\|^2 \\ &\quad +\eta_n(\eta_n - 2\delta_1)\|A_1 z_n - A_1 x^*\|^2) + 2\alpha_n M_4^2 \\ &\leq \|x_n - x^*\|^2 + (1 - \beta_n)\eta_n(\eta_n - 2\delta_1)\|A_1 z_n - A_1 x^*\|^2 + 2\alpha_n M_4^2. \end{aligned}$$

It follows from the condition (C1) that

$$\begin{aligned}(1 - b)c(2\delta_1 - d)\|A_1 z_n - A_1 x^*\|^2 &\leq \|x_n - x^*\|^2 - \|x_{n+1} - x^*\|^2 + 2\alpha_n M_4^2 \\ &\leq (\|x_n - x^*\| + \|x_{n+1} - x^*\|)\|x_n - x_{n+1}\| + 2\alpha_n M_4^2. \end{aligned}$$

It follows from (2.19) and the condition (C2) that

$$\lim_{n\to\infty} \|A_1 z_n - A_1 x^*\| = 0. \tag{2.22}$$

Next, we show that

$$\lim_{n\to\infty} \|A_2 u_n - A_2 x^*\| = 0. \tag{2.23}$$

From (2.20), we obtain that

$$\|x_{n+1} - x^*\|^2 \leq \beta_n\|x_n - x^*\|^2 + (1 - \beta_n)\|z_n - x^*\|^2 + 2\alpha_n M_4^2. \tag{2.24}$$

On the other hand, we have

$$\begin{aligned}\|z_n - x^*\|^2 &= \|P_C(u_n - \lambda_n A_2 u_n) - x^*\|^2 \\ &\leq \|(I - \lambda_n A_2)u_n - (I - \lambda_n A_2)x^*\|^2 \\ &= \|(u_n - x^*) - \lambda_n(A_2 u_n - A_u x^*)\|^2 \\ &= \|u_n - x^*\|^2 - 2\lambda_n\langle u_n - x^*, A_2 u_n - A_2 x^*\rangle + \lambda_n^2\|A_2 u_n - A_2 x^*\|^2 \\ &\leq \|u_n - x^*\|^2 - 2\lambda_n\delta_2\|A_2 u_n - A_2 x^*\|^2 + \lambda_n^2\|A_2 u_n - A_2 x^*\|^2 \\ &= \|u_n - x^*\|^2 + \lambda_n(\lambda_n - 2\delta_2)\|A_2 u_n - A_2 x^*\|^2. \end{aligned} \tag{2.25}$$

Substituting (2.25) into (2.24), we arrive at

$$\begin{aligned}\|x_{n+1}-x^*\|^2 &\le \beta_n\|x_n-x^*\|^2+(1-\beta_n)(\|u_n-x^*\|^2+\lambda_n(\lambda_n-2\delta_2)\|A_2u_n-A_2x^*\|^2)\\&\quad+2\alpha_n M_4^2\\&\le \beta_n\|x_n-x^*\|^2+(1-\beta_n)(\|x_n-x^*\|^2+\lambda_n(\lambda_n-2\delta_2)\|A_2u_n-A_2x^*\|^2)\\&\quad+2\alpha_n M_4^2\\&\le \|x_n-x^*\|^2+(1-\beta_n)\lambda_n(\lambda_n-2\delta_2)\|A_2u_n-A_2x^*\ |^2+2\alpha_n M_4^2.\end{aligned}$$

It follows from the condition (C1) that

$$\begin{aligned}(1-b)c'(2\delta_2-d')\|A_2u_n-A_2x^*\|^2 &\le \|x_n-x^*\|^2-\|x_{n+1}-x^*\|^2+2\alpha_n M_4^2\\&\le (\|x_n-x^*\|+\|x_{n+1}-x^*\|)\|x_n-x_{n+1}\|+2\alpha_n M_4^2.\end{aligned}$$

From (2.19) and the condition (C2), we see that (2.23) holds.

On the other hand, it follows from Lemma 1.2 that

$$\begin{aligned}\|u_n-x^*\|^2 &= \|T_{r_n}x_n-T_{r_n}x^*\|^2\\&\le \langle T_{r_n}x_n-T_{r_n}x^*, u_n-x^*\rangle\\&= \langle x_n-x^*, u_n-x^*\rangle\\&= \frac{1}{2}(\|u_n-x^*\|^2+\|x_n-x^*\|^2-\|u_n-x_n\|^2)\end{aligned}\tag{2.26}$$

and hence

$$\|u_n-x^*\|^2 \le \|x_n-x^*\|^2-\|x_n-u_n\|^2.\tag{2.27}$$

From (2.24), we have

$$\|x_{n+1}-x^*\|^2 \le \beta_n\|x_n-x^*\|^2+(1-\beta_n)\|u_n-x^*\|^2+2\alpha_n M_4^2.\tag{2.28}$$

Substituting (2.27) into (2.28), we arrive at

$$\begin{aligned}\|x_{n+1}-x^*\|^2 &\le \beta_n\|x_n-x^*\|^2+(1-\beta_n)(\|x_n-x^*\|^2-\|x_n-u_n\|^2)+2\alpha_n M_4^2\\&\le \|x_n-x^*\|^2-(1-\beta_n)\|x_n-u_n\|^2+2\alpha_n M_4^2\end{aligned}$$

It follows that

$$(1-\beta_n)\|x_n-u_n\|^2 \le (\|x_n-x^*\|+\|x_{n+1}-x^*\|)\|x_n-x_{n+1}\|+2\alpha_n M_4^2.$$

Thanks to the conditions (C1), (C2) and (2.19), we obtain that

$$\lim_{n\to\infty}\|x_n-u_n\|=0.\tag{2.29}$$

On the other hand, from (1.2), we have

$$\begin{aligned}\|z_n - x^*\|^2 &= \|P_C(I-\lambda_n A_2)u_n - P_C(I-\lambda_n A_2)x^*\|^2\\ &\le \langle (I-\lambda_n A_2)u_n - (I-\lambda_n A_2)x^*, z_n - x^*\rangle\\ &= \frac{1}{2}\Big\{\|(I-\lambda_n A_2)u_n - (I-\lambda_n A_2)x^*\|^2 + \|z_n - x^*\|^2\\ &\quad -\|(I-\lambda_n A_2)u_n - (I-\lambda_n A_2)x^* - (z_n - x^*)\|^2\Big\}\\ &\le \frac{1}{2}\Big\{\|u_n - x^*\|^2 + \|z_n - x^*\|^2 - \|u_n - z_n - \lambda_n(A_2u_n - A_2x^*)\|^2\Big\}\\ &= \frac{1}{2}\Big\{\|u_n - x^*\|^2 + \|z_n - x^*\|^2 - \|u_n - z_n\|^2\\ &\quad + 2\lambda_n\langle u_n - z_n, A_2u_n - A_2x^*\rangle - \lambda_n^2\|A_2u_n - A_2x^*\|^2\Big\},\end{aligned}$$

which implies that

$$\begin{aligned}\|z_n - x^*\|^2 &\le \|u_n - x^*\|^2 - \|u_n - z_n\|^2 + 2\lambda_n\langle u_n - z_n, A_2u_n - A_2x^*\rangle\\ &\quad - \lambda_n^2\|A_2u_n - A_2x^*\|^2\\ &\le \|x_n - x^*\|^2 - \|u_n - z_n\|^2 + 2\lambda_n\|u_n - z_n\|\|A_2u_n - A_2x^*\|. \end{aligned} \tag{2.30}$$

Substituting (2.30) into (2.24), we arrive at

$$\begin{aligned}\|x_{n+1} - x^*\|^2 &\le \beta_n\|x_n - x^*\|^2 + (1-\beta_n)(\|x_n - x^*\|^2 - \|u_n - z_n\|^2\\ &\quad + 2\lambda_n\|u_n - z_n\|A_2u_n - A_2x^*\|) + 2\alpha_n M_4^2\\ &\le \|x_n - x^*\|^2 - (1-\beta_n)\|u_n - z_n\|^2 + 2\lambda_n\|u_n - z_n\|A_2u_n - A_2x^*\|\\ &\quad + 2\alpha_n M_4^2,\end{aligned}$$

from which it follows that

$$\begin{aligned}(1-\beta_n)\|u_n - z_n\|^2 &\le \|x_n - x^*\|^2 - \|x_{n+1} - x^*\|^2 + 2\alpha_n M_4^2\\ &\quad + 2\lambda_n\|u_n - z_n\|A_2u_n - A_2x^*\|\\ &\le (\|x_n - x^*\| + \|x_{n+1} - x^*\|)\|x_n - x_{n+1}\| + 2\alpha_n M_4^2\\ &\quad + 2\lambda_n\|u_n - z_n\|A_2u_n - A_2x^*\|.\end{aligned}$$

It follows from the conditions (C1), (C2), (2.19) and (2.23) that

$$\lim_{n\to\infty}\|u_n - z_n\| = 0. \tag{2.31}$$

Next, in a similar way, we show that

$$\lim_{n\to\infty}\|y_n - z_n\| = 0. \tag{2.32}$$

From (1.2), we have

$$\begin{aligned}\|y_n - x^*\|^2 &= \|P_C(I-\eta_n A_1)z_n - P_C(I-\eta_n A_1)x^*\|^2\\ &\le \langle (I-\eta_n A_1)z_n - (I-\eta_n A_1)x^*, y_n - x^*\rangle\\ &= \frac{1}{2}\left\{\|(I-\eta_n A_1)z_n - (I-\eta_n A_1)x^*\|^2 + \|y_n - x^*\|^2\right.\\ &\quad \left. -\|(I-\eta_n A_1)z_n - (I-\eta_n A_1)x^* - (y_n - x^*)\|^2\right\}\\ &\le \frac{1}{2}\left\{\|z_n - x^*\|^2 + \|y_n - x^*\|^2 - \|z_n - y_n - \eta_n(A_1 z_n - A_1 x^*)\|^2\right\}\\ &\le \frac{1}{2}\left\{\|z_n - x^*\|^2 + \|y_n - x^*\|^2 - \|z_n - y_n\|^2\right.\\ &\quad \left. + 2\eta_n\langle z_n - y_n, A_1 z_n - A_1 x^*\rangle - \eta_n^2\|A_1 z_n - A_1 x^*\|^2\right\},\end{aligned}$$

which implies that

$$\begin{aligned}\|y_n - x^*\|^2 &\le \|z_n - x^*\|^2 - \|z_n - y_n\|^2 + 2\eta_n\langle z_n - y_n, A_1 z_n - A_1 x^*\rangle\\ &\quad -\eta_n^2\|A_1 z_n - A_1 x^*\|^2\\ &\le \|x_n - x^*\|^2 - \|z_n - y_n\|^2 + 2\eta_n\|z_n - y_n\|\|A_1 z_n - A_1 x^*\|. \qquad (2.33)\end{aligned}$$

Substituting (2.33) into (2.20), we arrive at

$$\begin{aligned}\|x_{n+1} - x^*\|^2 &\le \beta_n\|x_n - x^*\|^2 + (1-\beta_n)\left(\|x_n - x^*\|^2 - \|z_n - y_n\|^2\right.\\ &\quad \left. + 2\eta_n\|z_n - y_n\|\|A_1 z_n - A_1 x^*\|\right) + 2\alpha_n M_4^2\\ &\le \|x_n - x^*\|^2 - (1-\beta_n)\|z_n - y_n\|^2 + 2\eta_n\|z_n - y_n\|\|A_1 z_n - A_1 x^*\|\\ &\quad + 2\alpha_n M_4^2,\end{aligned}$$

from which it follows that

$$\begin{aligned}(1-\beta_n)\|z_n - y_n\|^2 &\le \|x_n - x^*\|^2 - \|x_{n+1} - x^*\|^2\\ &\quad + 2\eta_n\|z_n - y_n\|\|A_1 z_n - A_1 x^*\| + 2\alpha_n M_4^2\\ &\le (\|x_n - x^*\| + \|x_{n+1} - x^*\|)\|x_n - x_{n+1}\|\\ &\quad + 2\eta_n\|z_n - y_n\|\|A_1 z_n - A_1 x^*\|\\ &\quad + 2\alpha_n M_4^2.\end{aligned}$$

It follows from the conditions (C1), (C2), (2.19) and (2.22) that (2.32) holds.

On the other hand, from (2.1), we have

$$\begin{aligned}\|W_n y_n - x_n\| &\le \|x_n - x_{n+1}\| + \|x_{n+1} - W_n y_n\|\\ &\le \|x_n - x_{n+1}\| + \alpha_n\|\gamma f(x_n) - BW_n y_n\| + \beta_n\|x_n - W_n y_n\|.\end{aligned}$$

It follows that

$$(1-\beta_n)\|W_n y_n - x_n\| \le \|x_n - x_{n+1}\| + \alpha_n\|\gamma f(x_n) - BW_n y_n\|.$$

Since (2.19) and the conditions (C1), (C2), we obtain that

$$\lim_{n\to\infty}\|W_n y_n - x_n\| = 0. \qquad (2.34)$$

Notice that

$$\|W_n y_n - y_n\| \le \|y_n - z_n\| + \|z_n - u_n\| + \|u_n - x_n\| + \|x_n - W_n y_n\|.$$

From (2.29), (2.31), (2.32), and (2.34), we obtain that

$$\lim_{n\to\infty} \|W_n y_n - y_n\| = 0. \tag{2.35}$$

Next, we prove that

$$\limsup_{n\to\infty} \langle (\gamma f - B)z, x_n - z\rangle \le 0,$$

where $z = P_\Omega[I - (B - \gamma f)]z$. To see this, we choose a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that

$$\limsup_{n\to\infty} \langle (\gamma f - B)z, x_n - z\rangle = \lim_{i\to\infty} \langle (\gamma f - B)z, x_{n_i} - z\rangle. \tag{2.36}$$

Since $\{x_{n_i}\}$ is bounded, there exists a subsequence $\{x_{n_{i_j}}\}$ of $\{x_{n_i}\}$ which converges weakly to w. Without loss of generality, we can assume that $x_{n_i} \rightharpoonup w$. On the other hand, we have

$$\|x_n - y_n\| \le \|x_n - u_n\| + \|u_n - z_n\| + \|z_n - y_n\|.$$

It follows from (2.29), (2.31) and (2.32) that

$$\lim_{n\to\infty} \|x_n - y_n\| = 0. \tag{2.37}$$

Therefore, we see that $y_{n_i} \rightharpoonup w$. First, we prove that $w \in VI(C, A_1)$. For the purpose, let T be the maximal monotone mapping defined by (Δ):

$$Tx = \begin{cases} A_1 x + N_C x, & x \in C \\ \emptyset, & x \notin C. \end{cases}$$

For any given $(x, y) \in G(T)$, hence $y - A_1 x \in N_C$. Since $y_n \in C$, by the definition of N_C, we have

$$\langle x - y_n, y - A_1 x\rangle \ge 0. \tag{2.38}$$

On the other hand, from $y_n = P_C(I - \eta_n A_1)z_n$, we have

$$\langle x - y_n, y_n - (I - \eta_n A_1)z_n\rangle \ge 0$$

and hence

$$\left\langle x - y_n, \frac{y_n - z_n}{\eta_n} + A_1 z_n \right\rangle \ge 0.$$

From (2.38) and the δ_1-inverse monotonicity of A_1, we see that

$$\begin{aligned}
\langle x - y_{n_i}, y\rangle &\ge \langle x - y_{n_i}, A_1 x\rangle \\
&\ge \langle x - y_{n_i}, A_1 x\rangle - \left\langle x - y_{n_i}, \frac{y_{n_i} - z_{n_i}}{\eta_{n_i}} + A_1 z_{n_i} \right\rangle \\
&= \langle x - y_{n_i}, A_1 x - A_1 y_{n_i}\rangle + \langle x - y_{n_i}, A_1 y_{n_i} - A_1 z_{n_i}\rangle \\
&\quad - \left\langle x - y_{n_i}, \frac{y_{n_i} - z_{n_i}}{\eta_{n_i}} \right\rangle \\
&\ge \langle x - y_{n_i}, A_1 y_{n_i} - A_1 z_{n_i}\rangle - \left\langle x - y_{n_i}, \frac{y_{n_i} - z_{n_i}}{\eta_{n_i}} \right\rangle.
\end{aligned}$$

Since (2.32), $y_{n_i} \rightharpoonup w$ and A_1 is Lipschitz continuous, we obtain that

$$\lim_{i\to\infty} \langle x - y_{n_i}, y\rangle = \langle x - w, y\rangle \geq 0.$$

Notice that T is maximal monotone, hence $0 \in Tw$. This shows that $w \in VI(C, A_1)$. It follows from (2.32) and (2.37) that we also have

$$\lim_{n\to\infty} \|x_n - z_n\| = 0.$$

Therefore, we obtain that $z_{n_i} \rightharpoonup w$. Similarly, we can prove that $w \in VI(C, A_2)$. That is, $w \in VI = VI(C, A_2) \cap VI(C, A_1)$.

Next, we show that $w \in FP = \cap_{i=1}^{\infty} F(S_i)$. Suppose the contrary, $w \notin FP$, i.e., $Ww \neq w$. Since $y_{n_i} \rightharpoonup w$ and by the Opial condition, we see that

$$\begin{aligned}
\liminf_{i\to\infty} \|y_{n_i} - w\| &< \liminf_{i\to\infty} \|y_{n_i} - Ww\| \\
&\leq \liminf_{i\to\infty} \{\|y_{n_i} - Wy_{n_i}\| + \|Wy_{n_i} - Ww\|\} \\
&\leq \liminf_{i\to\infty} \{\|y_{n_i} - Wy_{n_i}\| + \|y_{n_i} - w\|\}. \qquad (2.39)
\end{aligned}$$

On the other hand, we have

$$\begin{aligned}
\|Wy_n - y_n\| &\leq \|Wy_n - W_n y_n\| + \|W_n y_n - y_n\| \\
&\leq \sup_{x\in K} \|Wx - W_n x\| + \|W_n y_n - y_n\|.
\end{aligned}$$

From Lemma 1.7 and (2.35), we obtain that $\lim_{n\to\infty} \|Wy_n - y_n\| = 0$, which combines with (2.39) yields that that

$$\liminf_{i\to\infty} \|y_{n_i} - w\| < \liminf_{i\to\infty} \|y_{n_i} - w\|,$$

which derives a contradiction. Thus, we have $w \in FP$.

Next, we show that $w \in EP(F)$. It follows from (2.29) that $u_n \rightharpoonup w$. Since $u_n = T_{r_n} x_n$, for any $y \in C$, we have

$$F(u_n, y) + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq 0.$$

From the condition (A2), we see that

$$\frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq F(y, u_n).$$

Replacing n by n_i, we arrive at

$$\left\langle y - u_{n_i}, \frac{u_{n_i} - x_{n_i}}{r_{n_i}}\right\rangle \geq F(y, u_{n_i}).$$

It follows from the condition (A4) that

$$F(y, w) \leq 0, \quad \forall y \in C.$$

For t with $0 < t \leq 1$ and $\rho \in C$, let $\rho_t = t\rho + (1-t)w$. Since $\rho \in C$ and $w \in C$, we have $\rho_t \in C$. By using the condition (A4), we see

$$0 = F(\rho_t, \rho_t) \leq tF(\rho_t, \rho) + (1-t)F(\rho_t, w) \leq tF(\rho_t, \rho).$$

which yields that

$$F(\rho_t, \rho) \geq 0.$$

By using the condition (A3), we see that $F(w, y) \geq 0, \forall y \in C$. This shows that $w \in EP(F)$. On the other hand, from (2.36), we see that

$$\limsup_{n\to\infty} \langle (\gamma f - B)z, x_n - z\rangle = \langle (\gamma f - B)z, w - z\rangle \leq 0. \tag{2.40}$$

Finally, we show that $x_n \to z$, as $n \to \infty$. From Lemma 1.3, we have

$$\begin{aligned}
\|x_{n+1} - z\|^2 &= \langle \alpha_n \gamma f(x_n) + \beta_n x_n + [(1-\beta_n)I - \alpha_n B]W_n y_n - z, x_{n+1} - z\rangle \\
&= \alpha_n\langle \gamma f(x_n) - Bz, x_{n+1} - z\rangle + \beta_n\langle x_n - z, x_{n+1} - z\rangle \\
&\quad + \langle [(1-\beta_n)I - \alpha_n B](W_n y_n - z), x_{n+1} - z\rangle \\
&\leq \alpha_n \gamma \langle f(x_n) - f(z), x_{n+1} - z\rangle + \alpha_n \langle \gamma f(z) - Bz, x_{n+1} - z\rangle \\
&\quad + \beta_n \|x_n - z\|\|x_{n+1} - z\| + (1 - \beta_n - \alpha_n\bar{\gamma})\|y_n - z\|\|x_{n+1} - z\| \\
&\leq \frac{\gamma\alpha}{2}\alpha_n(\|x_n - z\|^2 + \|x_{n+1} - z\|^2) + \alpha_n\langle \gamma f(z) - Bz, x_{n+1} - z\rangle \\
&\quad + (1 - \alpha_n\bar{\gamma})\|x_n - z\|\|x_{n+1} - z\| \\
&\leq \frac{\gamma\alpha}{2}\alpha_n(\|x_n - z\|^2 + \|x_{n+1} - z\|^2) + \alpha_n\langle \gamma f(z) - Bz, x_{n+1} - z\rangle \\
&\quad + \frac{(1 - \alpha_n\bar{\gamma})}{2}(\|x_n - z\|^2 + \|x_{n+1} - z\|^2) \\
&\leq \frac{1 - \alpha_n(\bar{\gamma} - \alpha\gamma)}{2}\|x_n - z\|^2 + \frac{1}{2}\|x_{n+1} - z\|^2 + \alpha_n\langle \gamma f(z) - Bz, x_{n+1} - z\rangle,
\end{aligned}$$

which implies that

$$\|x_{n+1} - z\|^2 \leq [1 - \alpha_n(\bar{\gamma} - \alpha\gamma)]\|x_n - z\|^2 + 2\alpha_n\langle \gamma f(z) - Bz, x_{n+1} - z\rangle.$$

From the condition (C2), (2.40) and using Lemma 1.4, we see that $\lim_{n\to\infty}\|x_n - z\| = 0$. This completes the proof. □

Remark 2.2 Theorem 2.1 improves Theorem 3.1 of Chang et al. [5] as a special case. To be more precise, we consider a pair of inverse-strongly monotone mappings instead of a single mapping based on the extragradient-like method. The common element z which is the optimality condition for the minimization problem $\min_{x\in\Omega} \frac{1}{2}\langle Bx, x\rangle - h(x)$, where h is the potential function for γf is more meaningful; see [11,28,30].

Letting $\gamma = 1$ and $B = I$, the identity mapping, we can obtain the following result immediately from Theorem 2.1.

Corollary 2.3 *Let C be a nonempty closed convex subset of a Hilbert space H and F a bifunction from $C \times C$ to $\mathbb{R}$ which satisfies (A1)–(A4). Let A_j be a δ_j-inverse-strongly monotone mapping of C into H for each $j = 1, 2$, $\{S_i : C \to C\}$ a family of infinitely nonexpansive mappings with $\Omega := FP \cap EP(F) \cap VI \neq \emptyset$, where $FP = \cap_{i=1}^{\infty} F(S_i)$ and $VI = VI(C, A_1) \cap VI(C, A_2)$. Let f be a contraction of C into itself with the contractive coefficient $\alpha (0 < \alpha < 1)$. Let $\{x_n\}$ be a sequence generated by $x_1 \in C$ and*

$$\begin{cases}
F(u_n, y) + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq 0, \quad \forall y \in C, \\
z_n = P_C(u_n - \lambda_n A_2 u_n), \\
y_n = P_C(z_n - \eta_n A_1 z_n), \\
x_{n+1} = \alpha_n f(x_n) + \beta_n x_n + (1 - \beta_n - \alpha_n)W_n y_n, \quad \forall n \geq 1,
\end{cases}$$

where $\{W_n : C \to C\}$ *is the sequence defined by* (1.7)*,* $\{\alpha_n\}$ *and* $\{\beta_n\}$ *are sequences in* $[0, 1]$ *and* $\{r_n\}$, $\{\lambda_n\}$ *and* $\{\eta_n\}$ *are positive number sequences. Assume that the above control sequences satisfy the following restrictions*

(C1) $0 < a \le \beta_n \le b < 1, 0 < c \le \eta_n \le d < 2\delta_1, 0 < c' \le \lambda_n \le d' < 2\delta_2, \forall n \ge 1;$
(C2) $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=1}^{\infty} \alpha_n = \infty$*;*
(C3) $\liminf_{n\to\infty} r_n > 0, \sum_{n=1}^{\infty} |r_{n+1} - r_n| < \infty$*;*
(C4) $\lim_{n\to\infty}(\lambda_n - \lambda_{n+1}) = \lim_{n\to\infty}(\eta_n - \eta_{n+1}) = 0.$

Then $\{x_n\}$ *converges strongly to* $z \in \Omega$*, where* $z = P_\Omega f(z)$*.*

Remark 2.4 If $A_1 = A_2$ and $\lambda_n = \eta_n$ for each $n \ge 1$, then Corollary 2.2 is reduced to Theorem 3.1 of Chang et al. [5].

If $A_1 = A_2 = 0$, then Theorem 2.1 is reduced to the following result.

Corollary 2.5 *Let* C *be a nonempty closed convex subset of a Hilbert space* H *and* F *a bifunction from* $C \times C$ *to* $\mathbb{R}$ *which satisfies* $(A1)$*–*$(A4)$*. Let* $\{S_i : C \to C\}$ *be a family of infinitely nonexpansive mappings with* $\Omega := F \cap EP(F) \ne \emptyset$*, where* $F = \cap_{i=1}^{\infty} F(S_i)$*. Let* f *be a contraction of* H *into itself with the contractive coefficient* $\alpha(0 < \alpha < 1)$, $B : C \to H$ *a strongly positive linear bounded self-joint operator with the coefficient* $\bar{\gamma} > 0$*. Assume that* $0 < \gamma < \bar{\gamma}/\alpha$*. Let* $\{x_n\}$ *be a sequence generated by* $x_1 \in H$ *and*

$$\begin{cases} F(u_n, y) + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \ge 0, \quad \forall y \in C, \\ x_{n+1} = \alpha_n \gamma f(x_n) + \beta_n x_n + [(1 - \beta_n)I - \alpha_n B]W_n u_n, \quad \forall n \ge 1, \end{cases}$$

where $\{W_n : C \to C\}$ *is the sequence defined by* (1.7)*,* $\{\alpha_n\}$ *and* $\{\beta_n\}$ *are sequences in* $[0, 1]$ *and* $\{r_n\}$ *is a positive number sequence. Assume that the above control sequences satisfy the following restrictions*

(C1) $0 < a \le \beta_n \le b < 1, \forall n \ge 1;$
(C2) $\lim_{n\to\infty} \alpha_n = 0$*, and* $\sum_{n=1}^{\infty} \alpha_n = \infty$*;*
(C3) $\liminf_{n\to\infty} r_n > 0, \lim_{n\to\infty} r_n/r_{n+1} = 1.$

Then $\{x_n\}$ *converges strongly to* $z \in \Omega$*, where* $z = P_\Omega(I - B + \gamma f)z$*.*

Remark 2.6 Corollary 2.5 improves the results of Colao, Marino and Xu [6] from a finite family of nonexpansive mappings to an infinite family of nonexpansive mappings.

If $\gamma = 1$, $B = I$, the identity mapping, $f(x) \equiv u \in C$, for all $x \in H$, and $A_1 = A_2 = A$, then Theorem 2.1 is reduced to the following result.

Corollary 2.7 *Let* C *be a nonempty closed convex subset of a Hilbert space* H *and* F *a bifunction from* $C \times C$ *to* $\mathbb{R}$ *which satisfies* $(A1)$*–*$(A4)$*. Let* A *be a* δ*-inverse-strongly monotone mapping of* C *into* H, $\{S_i : C \to C\}$ *a family of infinitely nonexpansive mappings with* $\Omega := FP \cap EP(F) \cap VI(C, A) \ne \emptyset$*, where* $FP = \cap_{i=1}^{\infty} F(S_i)$*. Let* $\{x_n\}$ *be a sequence generated by* $x_1 = u \in C$ *and*

$$\begin{cases} F(u_n, y) + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \ge 0, \quad \forall y \in C, \\ z_n = P_C(u_n - \lambda_n A u_n), \\ x_{n+1} = \alpha_n u + \beta_n x_n + (1 - \beta_n - \alpha_n) W_n P_C(z_n - \lambda_n A z_n), \quad \forall n \ge 1, \end{cases}$$

where $\{W_n : C \to C\}$ *is the sequence defined by* (1.7)*,* $\{\alpha_n\}$ *and* $\{\beta_n\}$ *are sequences in* $[0, 1]$ *and* $\{r_n\}$ *and* $\{\lambda_n\}$ *are positive number sequences. Assume that the above control sequences satisfy the following restrictions*

(C1) $0 < a \le \beta_n \le b < 1, 0 < c \le \lambda_n \le d < 2\delta, \forall n \ge 1$;
(C2) $\lim_{n\to\infty} \alpha_n = 0$, *and* $\sum_{n=1}^{\infty} \alpha_n = \infty$;
(C3) $\liminf_{n\to\infty} r_n > 0, \sum_{n=1}^{\infty} |r_{n+1} - r_n| < \infty$;
(C4) $\lim_{n\to\infty}(\lambda_n - \lambda_{n+1}) = 0$.

Then $\{x_n\}$ *converges strongly to* $z \in \Omega$, *where* $z = P_\Omega u$.

Remark 2.8 Corollary 2.7 improves the results of Plubtieng and Punpaeng [13] from a single nonexpansive mapping to an infinite family of nonexpansive mappings.

Letting $F(x, y) \equiv 0$, for $\forall x, y \in C$ and $\{r_n\} = 1, \forall n \ge 1$, in Theorem 2.1, we have $u_n = P_C x_n$. Then the following results can be obtained immediately.

Corollary 2.9 *Let* C *be a nonempty closed convex subset of a Hilbert space* H. *Let* A_j *be a* δ_j*-inverse-strongly monotone mapping of* C *into* H *for each* $j = 1, 2$, $\{S_i : C \to C\}$ *a family of infinitely nonexpansive mappings with* $\Omega := FP \cap VI \ne \emptyset$, *where* $FP = \cap_{i=1}^{\infty} F(S_i)$ *and* $VI = VI(C, A_1) \cap VI(C, A_2)$. *Let* f *be a contraction of* H *into itself with the contractive coefficient* $\alpha (0 < \alpha < 1)$, $B : C \to H$ *a strongly positive linear bounded self-joint operator with the coefficient* $\bar{\gamma} > 0$. *Assume that* $0 < \gamma < \bar{\gamma}/\alpha$. *Let* $\{x_n\}$ *be a sequence generated by* $x_1 \in H$ *and*

$$\begin{cases} z_n = P_C(I - \lambda_n A_2) P_C x_n, \\ y_n = P_C(z_n - \eta_n A_1 z_n), \\ x_{n+1} = \alpha_n \gamma f(x_n) + \beta_n x_n + [(1 - \beta_n) I - \alpha_n B] W_n y_n, \quad \forall n \ge 1, \end{cases}$$

where $\{W_n : C \to C\}$ *is the sequence defined by* (1.7), $\{\alpha_n\}$ *and* $\{\beta_n\}$ *are sequences in* $[0, 1]$ *and* $\{\lambda_n\}$ *and* $\{\eta_n\}$ *are positive number sequences. Assume that the above control sequences satisfy the following restrictions*

(C1) $0 < a \le \beta_n \le b < 1, 0 < c \le \eta_n \le d < 2\delta_1, 0 < c' \le \lambda_n \le d' < 2\delta_2, \forall n \ge 1$;
(C2) $\lim_{n\to\infty} \alpha_n = 0$, *and* $\sum_{n=1}^{\infty} \alpha_n = \infty$;
(C3) $\lim_{n\to\infty}(\lambda_n - \lambda_{n+1}) = \lim_{n\to\infty}(\eta_n - \eta_{n+1}) = 0$.

Then $\{x_n\}$ *converges strongly to* $z \in \Omega$, *where* $z = P_\Omega(I - B + \gamma f)z$.

Further, putting $f(x) = u \in C$, for all $x \in H, \gamma = 1$ and $B = I$, the identity mapping, we can obtain the following easily from Corollary 2.9.

Corollary 2.10 *Let* C *be a nonempty closed convex subset of a Hilbert space* H. *Let* A_j *be a* δ_j*-inverse-strongly monotone mapping of* C *into* H *for each* $j = 1, 2$, $\{S_i : C \to C\}$ *a family of infinitely nonexpansive mappings with* $\Omega := FP \cap VI \ne \emptyset$, *where* $FP = \cap_{i=1}^{\infty} F(S_i)$ *and* $VI = VI(C, A_1) \cap VI(C, A_2)$. *Let* $\{x_n\}$ *be a sequence generated by* $x_1 = u \in C$ *and*

$$\begin{cases} z_n = P_C(x_n - \lambda_n A_2 x_n), \\ y_n = P_C(z_n - \eta_n A_1 z_n), \\ x_{n+1} = \alpha_n u + \beta_n x_n + (1 - \beta_n - \alpha_n) W_n y_n, \ \forall n \ge 1, \end{cases}$$

where $\{W_n : C \to C\}$ *is the sequence defined by* (1.7), $\{\alpha_n\}$ *and* $\{\beta_n\}$ *are sequences in* $[0, 1]$ *and* $\{\lambda_n\}$ *and* $\{\eta_n\}$ *are positive number sequences. If the above control sequences satisfy the following restrictions*

(C1) $0 < a \le \beta_n \le b < 1, 0 < c \le \eta_n \le d < 2\delta_1, 0 < c' \le \lambda_n \le d' < 2\delta_2, \forall n \ge 1$;
(C2) $\lim_{n\to\infty} \alpha_n = 0$, *and* $\sum_{n=1}^{\infty} \alpha_n = \infty$;
(C3) $\lim_{n\to\infty}(\lambda_n - \lambda_{n+1}) = \lim_{n\to\infty}(\eta_n - \eta_{n+1}) = 0$.

Then $\{x_n\}$ *converges strongly to* $z = P_\Omega u$.

Remark 2.11 Corollary 2.10 includes Ceng and Yao [4], Yao and Yao [30] as special cases.

3 Applications

As applications of our main results, we can obtain the following results.

First, we consider another class of important nonlinear operators: strict pseudo-contractions.

Recall that a mapping $S : C \to C$ is said to be a κ-strict pseudo-contraction if there exists a constant $\kappa \in [0, 1)$ such that

$$\|Sx - Sy\|^2 \le \|x - y\|^2 + \kappa\|(I - S)x - (I - S)y\|^2, \quad \forall x, y \in C.$$

Note that the class of κ-strict pseudo-contractions strictly includes the class of nonexpansive mappings.

Put $A = I - S$, where $S : C \to C$ is a κ-strict pseudo-contraction. Then A is $\frac{1-\kappa}{2}$-inverse-strongly monotone; see [1,5] for more details.

Theorem 3.1 *Let C be a nonempty closed convex subset of a Hilbert space H and F a bifunction from $C \times C$ to $\mathbb{R}$ which satisfies (A1)–(A4). Let T_j be a κ_j-strict pseudo-contractive mapping of C into itself for each $j = 1, 2$, $\{S_i : C \to C\}$ a family of infinitely nonexpansive mappings with $\Omega := FP \cap EP(F) \cap VI \neq \emptyset$, where $FP = \cap_{i=1}^{\infty} F(S_i)$ and $VI = VI(C, A_1) \cap VI(C, A_2)$. Let f be a contraction of H into itself with the contractive coefficient $\alpha (0 < \alpha < 1)$, $B : C \to H$ a strongly positive linear bounded self-joint operator with the coefficient $\bar{\gamma} > 0$. Assume that $0 < \gamma < \bar{\gamma}/\alpha$. Let $\{x_n\}$ be a sequence generated by $x_1 \in H$ and*

$$\begin{cases} F(u_n, y) + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \ge 0, \quad \forall y \in C, \\ z_n = (1 - \lambda_n)u_n + \lambda_n T_2 u_n, \\ y_n = (1 - \eta_n)z_n + \eta_n T_1 z_n, \\ x_{n+1} = \alpha_n \gamma f(x_n) + \beta_n x_n + [(1 - \beta_n)I - \alpha_n B]W_n y_n, \quad \forall n \ge 1, \end{cases}$$

where $\{W_n : C \to C\}$ is the sequence defined by (1.7), $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in $[0, 1]$ and $\{r_n\}$, $\{\lambda_n\}$ and $\{\eta_n\}$ are positive number sequences. Assume that the above control sequences satisfy the following restrictions

(C1) $0 < a \le \beta_n \le b < 1, 0 < c \le \eta_n \le d < 2\delta_1, 0 < c' \le \lambda_n \le d' < 2\delta_2, \forall n \ge 1$;
(C2) $\lim_{n\to\infty} \alpha_n = 0$, *and* $\sum_{n=1}^{\infty} \alpha_n = \infty$;
(C3) $\liminf_{n\to\infty} r_n > 0, \lim_{n\to\infty} r_n/r_{n+1} = 1$;
(C4) $\lim_{n\to\infty}(\lambda_n - \lambda_{n+1}) = \lim_{n\to\infty}(\eta_n - \eta_{n+1}) = 0$.

Then $\{x_n\}$ converges strongly to $z \in \Omega$, where $z = P_\Omega(I - B + \gamma f)z$.

Proof Taking $A_j = I - T_j : C \to H$, wee see that $A_j : C \to H$ is δ_j-strict pseudo-contraction with $\delta_j = \frac{1-\kappa_j}{2}$ and $F(T_j) = VI(C, A_j)$ for $j = 1, 2$. We can obtain the desired conclusion easily from Theorem 2.1. This completes the proof. □

Remark 3.2 Theorem 3.1 mainly improves Theorem 4.3 of Chang et al. [5], Theorem 4.2 of Plubtieng and Punpaeng [13] and Theorem 3.3 of Yao and Yao [30], respectively.

Second, we utilize the results presented in the paper to study the following optimization problem:

$$\min_{x\in C} h(x), \tag{3.1}$$

where C is a nonempty bounded closed convex subset of a real Hilbert space, and $h : C \to \mathbb{R}$ is a convex and lower semi-continuous functional. We denote by $\mathcal{H}$ the set of solutions of

(3.1). Let $F : C \times C \to \mathbb{R}$ be a bifunction defined by $F(x, y) = h(y) - h(x)$. We consider the following equilibrium problem, that is to find $x \in C$ such that

$$F(x, y) \geq 0, \quad \forall y \in C. \tag{3.2}$$

It is easy to see that the bifunction F satisfies conditions $(A1)$–$(A4)$ and $EP(F) = \mathcal{H}$, where $EP(F)$ is the set of solutions of equilibrium problem (3.2).

Theorem 3.3 *Let C be a nonempty bounded closed convex subset of a Hilbert space H and h be defined as the above. Let f be a contraction of H into itself with the contractive coefficient $\alpha(0 < \alpha < 1)$, $B : C \to H$ a strongly positive linear bounded self-joint operator with the coefficient $\bar{\gamma} > 0$. Assume that $0 < \gamma < \bar{\gamma}/\alpha$. Let $\{x_n\}$ be a sequence generated by $x_1 \in H$ and*

$$\begin{cases} h(y) - h(u_n) + \frac{1}{r_n}\langle y - u_n, u_n - x_n\rangle \geq 0, & \forall y \in C, \\ x_{n+1} = \alpha_n \gamma f(x_n) + \beta_n x_n + [(1 - \beta_n)I - \alpha_n B]u_n, & \forall n \geq 1, \end{cases}$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in $[0, 1]$ and $\{r_n\}$ is a positive number sequence. Assume that the above control sequences satisfy the following restrictions

(C1) $0 < a \leq \beta_n \leq b < 1, \forall n \geq 1$;
(C2) $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=1}^{\infty} \alpha_n = \infty$*;*
(C3) $\liminf_{n\to\infty} r_n > 0, \lim_{n\to\infty} r_n/r_{n+1} = 1$.

Then $\{x_n\}$ converges strongly to $z \in \mathcal{H}$, where $z = P_{\mathcal{H}}(I - B + \gamma f)z$.

Proof Letting $S_i = I$, the identity mapping, for $\forall i \geq 1$, we see that $W_n = I$. Taking $A_1 = A_2 = 0$, we can conclude the desired conclusion easily from Theorem 2.1. □

Remark 3.4 Theorem 3.3 includes Theorem 4.1 of Chang et al. [5] as a special case.

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